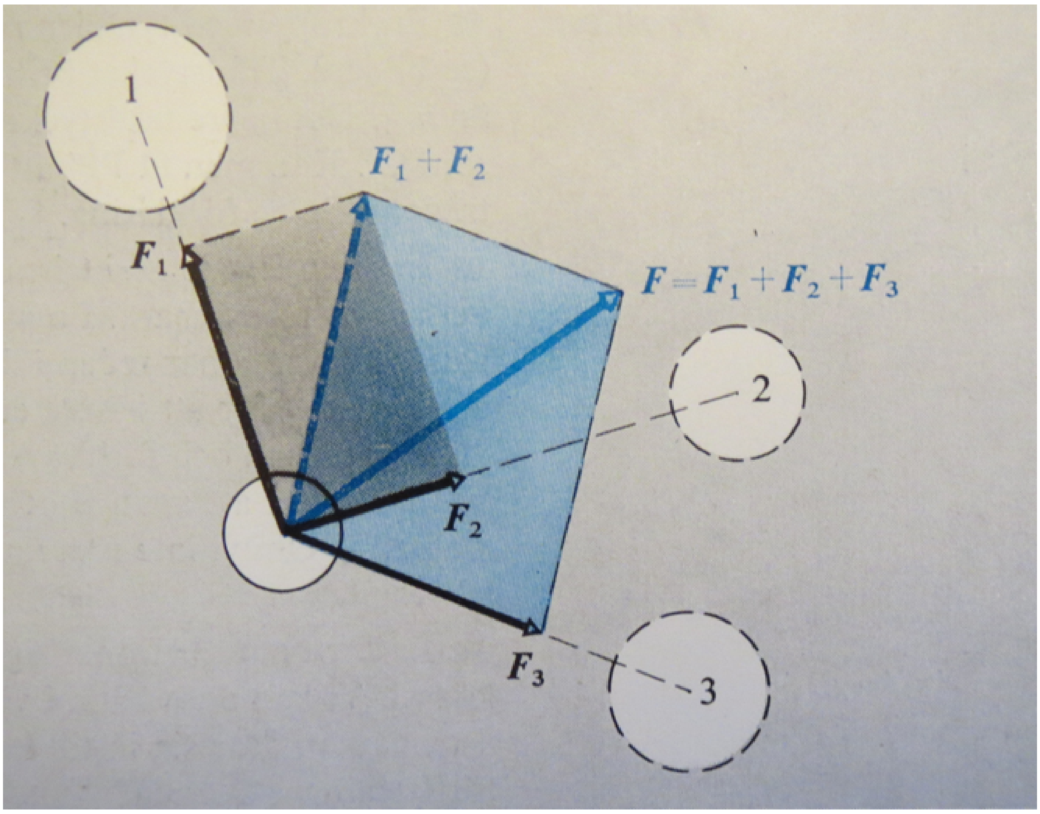
Newtons laws - Equations of Motion - Problems

John V Petersen



Newtons laws

art-science-soul

Contents

1. **In this article Newtons Laws are defined, explicit Equations of motion are deduced and problems are given and Solutions to the Problems.**

First we will look at Newtons Laws

Newtons 1. Law (the law of inertia)
A body in uniform motion remains in uniform motion, and a body at rest remains at rest, when the net force acting on the body is zero ($\boldsymbol{F}_{res} = \boldsymbol{0}$).

Motion is always measured with respect to a reference frame. A reference frame in which the Newton's 1 st law is valid is called as inertial reference frame . We may say that **Newton's 1 st law defines inertial reference frame.**

$$\boldsymbol{F}_{res} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{v} = \text{konstant} \quad \Rightarrow \quad \boldsymbol{a} = \boldsymbol{0}$$

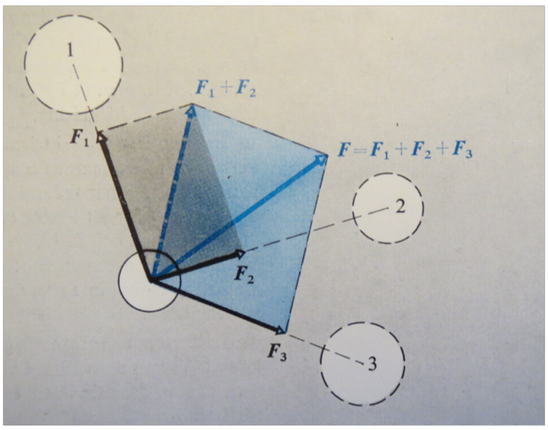
Newtons 2. Law
$\boldsymbol{F}_{res}$ (the net force acting on the body) is proportional , to the product of its mass multiplying its acceleration,

$$\boldsymbol{F}_{res} = m \cdot \boldsymbol{a} \quad \text{or,} \quad \boldsymbol{F}_{res} = m \cdot \frac{d\,\boldsymbol{v}}{dt}$$

Newtons 2. law is **valid only** in an inertial reference frame (i.e., where the acceleration is measured.)

Newtons 3. Law (the law of action and reaction):
If a given body A acts on another body B with a force, then B will also act on A with a force equal in magnitude but opposite in direction.

Newtons 4. Law The Superposition Principle
If several forces act on a body, then the resultant acceleration of the body is the sum of the acceleration computed for each of the forces. It implies vectoraddition of the forces acting on the body



the resultant force: $\boldsymbol{F}$

2. Integration of the Equation of Motion (Newtons 2. law) and those explicit Equations of Motion

A particle of mass m moves along the x-axis of a coordinate system under the action of a constant force $K = |\boldsymbol{K}|$ in the positive x-direction.
Analyze the motion; i.e., find the position of the particle as a function of time.

The *equation of motion* is obtained by inserting the force into Newton's second law:

[redacted]

Integration of (2.3) gives

$$v = \int \frac{K}{m} \, \mathrm{d}t \, ,$$

or

$$v(t) = \frac{K}{m} t + c_1 \, , \qquad (2.4)$$

c_1 is a constant of integration.

Equation (2.4) can be written in the form $$\frac{dx}{dt} = \frac{K}{m}\,t + c_1 \qquad (2.5)$$

Again we have a separable differential equation

[redacted]

$$x_0 = 0 + 0 + c_2 \quad , \quad c_2 = x_0 \qquad (2.7)$$

The complete solution is therefore:

$$x = \frac{1}{2}\,\frac{K}{m}\,t^2 + v_0 t + x_0 \qquad (2.8)$$

If the initial time is $t = t_0$ rather than $t = 0$, we get

$$v_0 = \frac{K}{m}\, t_0 + c_1 \qquad , \quad c_1 = v_0 - \frac{K}{m}\, t_0 \qquad (2.4)$$

$$x_0 = \frac{1}{2}\,\frac{K}{m}\, t_0^{\,2} + \left(v_0 - \frac{K}{m}\, t_0 \right)\cdot t + c_2 \qquad (2.7)$$

[redacted]

We get

$$x = \frac{1}{2}\,\frac{K}{m}\,\left(t - t_0 \right)^2 + v_0 \cdot \left(t - t_0 \right) + x_0 \qquad (2.9)$$

This exsample applies to motions like the free fall (in vacuum) near the surface of the earth, if the total vertical extension is so small that the variation with height of the gravitational force can be disregarded.

Now we return to equation (2.8). At the time $t = 0$ the velocity of the particle is v_0 and the position is x_0 .

$$x = \frac{1}{2}\,\frac{K}{m}\,t^2 + v_0 t + x_0 \qquad (2.8)$$

[redacted]

$$2\cdot\left(\frac{K}{m}\right)\cdot\left(x - x_0\right) = \left(\frac{K}{m}\right)^2\cdot\left(\frac{v - v_0}{\left(\frac{K}{m}\right)}\right)^2 + 2\cdot v_0\cdot\left(\frac{K}{m}\right)\cdot\left(\frac{v - v_0}{\left(\frac{K}{m}\right)}\right) \qquad \Leftrightarrow$$

$$\left(v - v_0\right)^2 + 2\cdot v_0\cdot\left(v - v_0\right) = 2\cdot\left(\frac{K}{m}\right)\cdot\left(x - x_0\right) \qquad \Leftrightarrow$$

[redacted]

$$v^2 - {v_0}^2 = 2 \cdot \left(\frac{K}{m} \right) \cdot (x - x_0) \qquad (2.11)$$

i.e. at the time $t = 0$, the velocity v_0 and the position x_0 , we *get*:

$$x = \frac{1}{2} \left(\frac{K}{m} \right) t^2 + v_0 t + x_0 \qquad (2.8)$$

$$v = \left(\frac{K}{m} \right) t + v_0 \qquad (2.10)$$

$$v^2 - {v_0}^2 = 2 \cdot \left(\frac{K}{m} \right) \cdot (x - x_0) \qquad (2.11)$$

The force K and the mass m are constante, then $\frac{K}{m} = a$, is a constant acceleration.

If we substitute $a = \frac{K}{m}$ in the equations above, then we precisely got the kinematic equations of motion known from school.

You only look at the movements, in the first instance, and don't think about the forces, i.e. the cause of the movements.

3. Problems
Below you get four problems. Use the equations (2.3) to (2.11) to solve them.
The difficulty level of the problems slightly increases , i.e. problem 1 and 2, are the easiest.

problem 1

The speed of a train is reduced uniformly from $15 \, \frac{m}{s}$ to $7.0 \, \frac{m}{s}$, while traveling a distance of $90 \, m$.

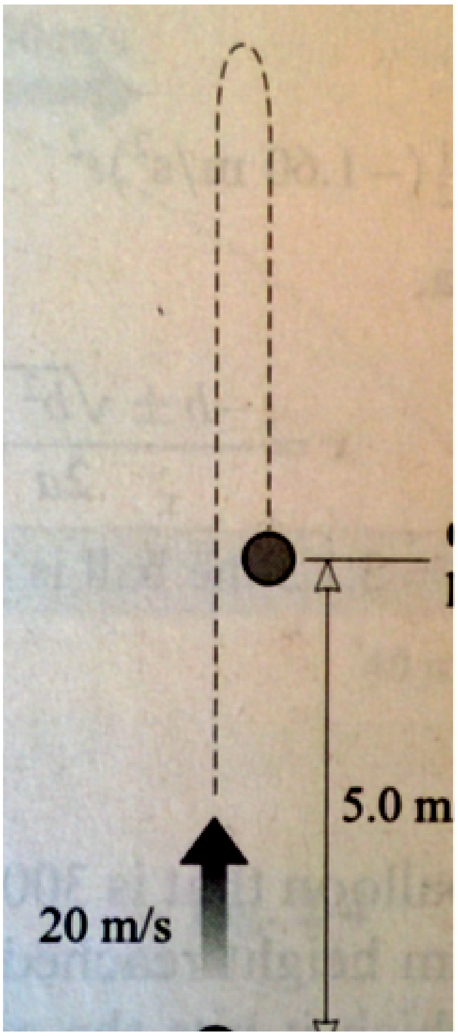
a) Compute the acceleration a.

b) How much farther will the train travel before coming to rest, provided the acceleration remains constant ?

problem 2

A stone is thrown straight upward, in the gavitationel filed of the earth, with a speed of $20\ \frac{m}{s}$ at initial time $t = 0$.
It is caught on its way down at a point $5.0\ m$ above where it was thrown (fig. 1). Disregard air resistance.

(fig. 1)



a) How fast was it going, when it was caught ?

b) How long did the trip take ?

problem 3

A locomotive is driving with a velocity $v_0 = 100 \dfrac{km}{h}$ on a straight railroad track.
The mass M of the engine is 10 tons. At the time $t=0$ the engine begins to brake by blocking the wheels.

This causes a frictional force F to appear. We assume F to be constant.

a) Write the aquation of motion for the train at times $t > 0$

b) Suppose the train comes to a halt after 400 *m*. Determine the magnitude of F and the time t_1, at which the train stops.

problem 4

A stone is thrown vertically upwards in the gravitational field of the earth. A point P is at height h above the starting point of the stone.
The stone passes P on its way upward, 2 seconds after it was thrown, and 4 seconds after it was thrown the stone passes P on its downward fall.

a) Calculate h and the initial velocity v_0 of the stone. Disregard air resistance.

Solutions to the problems

problem 1

a) Let us take the direction of motion to be the positive x-direction.

[redacted]

$$v = \pm\sqrt{{v_0}^2 + 2\cdot g\cdot y} = \pm\sqrt{\left(20\ \frac{m}{s}\right)^2 - 2\cdot 9.82\ \frac{m}{s^2}\cdot 5.0\ m}$$

$$= \pm 17.3724 \frac{m}{s} \approx \pm 17 \frac{m}{s}$$

$v = -17 \frac{m}{s}$. We take the negative sign because the stone is moving downward, in the negative di[redacted]

$$v = 0 = -\left(\frac{F}{M}\right) \cdot t_1 + v_0$$

i.e. two equations with two unknown variables $\left(t_1\,,\ F\right)$

$$\frac{F}{M}\cdot t_1 \;=\; v_0 \quad\Leftrightarrow\quad F \;=\; \frac{M\cdot v_0}{t_1} \qquad \textit{we}\ \text{substitute into (2.8):}$$

[redacted]

We substitute t_1 and t_2 into (i) :

$$h \;=\; \frac{1}{2}\cdot g\cdot t_1{}^2 \;+\; v_0\cdot t_1$$

$$h = \frac{1}{2} \cdot g \cdot t_2^{\,2} + v_0 \cdot t_2$$

i.e. two equations with two unknown variables (v_0 , h)

[redacted]